ON SOME:
(alpha, alpha star, alpha star star) continuous function

Nassir Ali Zubain
Wasit University
Wasit, Iraq
E-mail: nasseeral480@gmail.com
52001

Ali Khalaf Hussain
Wasit University
Wasit, Iraq
alhachamia@uowasit.edu.iq
52001

ABSTRACT. In this paper a new class of functions, such that semi $\alpha$–continuous function is introduced for topological spaces, and the second type is semi $\alpha^*$–continuous function and three class is semi $\alpha^{**}$–continuous function. We have taken in our study this continuous homeomorphism function (bijective = injective + surjective), we introduce using practical examples of mathematical formulas and considering them as a direct application to the validity of the observations. We also, study the relationship between these concepts that we referred to at the beginning of the research.

Keywords: $\alpha$-continuous function, $\alpha^*$-continuous function, $\alpha^{**}$-continuous function, $\alpha$-open.

Introduction

T.Noiri.[1] introduce the concept of new ($\alpha$-continuous function), according to the source we got" On –continuous function". And when we return to the basic concepts, we studied with Y.Yousif, R. Mejed. [3]. And introduce semi-open by N. Levine,[4] "Semi-open sets and semi-continuity in Topological space", also N. Levine, [5] "Generalized Closed Sets in Topology Rend.". So we have an important concept to contest the results with the previous concepts O. Njastad. [6] "On Some Classes of Nearly Open Sets" we studied some characteristics of these functions by using definitions Also, we introduce using practical examples of mathematical formulas and considering them as a direct application to the validity of the observations. We also, study the relationship between these concepts that we referred to at the beginning of the research.

Definition 1.1. [4]
A topological space $(X, \tau)$, and $M \subseteq X$ be named semi-open set. exists open set $O$, then $O \subseteq M \subseteq Cl O$.

Theorem 1.2. [4]
A topological space \((X, \tau), M \subseteq X\), and \(M\) be semi-open set if and only if \(M \subseteq \text{Cl Int } M\).

**Proof:**
If \(M\) is a semi-open set, and \(O\) is open set.
By defined \(O \subset M \subset \text{Cl } O\).
Then \(\text{Int } M \subset M\). Hence \(O \subset \text{Int } M \subset M \subset \text{Cl } O\),
Thus \(\text{Cl } O \subset \text{Cl Int } M \subset \text{Cl } M \subset \text{Cl } O\), \((\text{Cl } Cl O = \text{Cl } O)"
So \(M \subset \text{Cl } O\). Therefore \(M \subset \text{Cl Int } M\).

**On the other hand,** \(M\) is a semi-open set. Hence \(M \subset \text{Cl Int } M\).
Then \(\text{Int } M \subset M\). So \(\text{Int } M \subset M \subset \text{Cl } \text{Int } M\).
Therefore \(M\) it is semi-open set.

**Example 1.3.**
If \(X = \{4,1,8\}, T = \{\emptyset, \{4\}, \{1\}, \{4,1\}, X\}\), a topology defined on \(X\).
Thus \(\{4,8\}, \{8,1\}\) are semi-open.
However \(\{4,8\} \cap \{8,1\} = \{8\}\), it is not semi-open set.

**Definition 1.4.** [5]
a topological space \((X, \tau)\), too \(M \subseteq X\), be called semi-closed set, then a closed set \(C\) such that
\[\text{Int } C \subset M \subset C.\]

**Theorem 1.5.** [5]
a topological space \((X, \tau)\), then \(M \subseteq X\), \(M\) be semi-closed set, iff \(\text{Int } \text{Cl } M \subset M\).

**Proof:**
If \(M\) is a semi-closed set. Thus \(M^C\) is semi-open set.
So \(M^C \subset \text{Cl } \text{Int } M^C\). Then \(\text{Int } \text{Cl } M \subset M\).
Conversely, \(M\) is semi-closed set. Therefore \(\text{Int } \text{Cl } M \subset M\).
Wherefore \(M \subset \text{Cl } M\). Hence, \(\text{Int } \text{Cl } M \subset M \subset \text{Cl } M\) and \(\text{Cl } M\) is closed set.
As a result, \(M\) is semi-closed set

**Corollary 1.6.** [4]
The intersection of two semi-closed sets is semi-closed set in any topological space.

**Proof:**
Let \(N\) and \(F\) two semi-closed sets in \((X, \tau)\) be a topological space. then by definition semi-closed we get \(\text{Int } \text{Cl } N \subset N\), and \(\text{Int } \text{Cl } F\), therefore \(\text{Int } \text{Cl } N \cap \text{Int } \text{Cl } F \subset N \cap F\), but \(\text{Int } \text{Cl } N \cap \text{Int } \text{Cl } F = \text{Int } (\text{Cl } N \cap \text{Cl } F)\), then \(\text{Int } (\text{Cl } N \cap \text{Cl } F) \subset N \cap F\) and
\(\text{Int } (\text{Cl } N \cap \text{Cl } F) \subset \text{Int } (\text{Cl } N \cap \text{Cl } F)\), so \(\text{Cl } (N \cap F) \subset \text{Cl } N \cap \text{Cl } F\)
Thus \(\text{Int } (\text{Cl } (N \cap F)) \subset N \cap F\),
And from it, we find that \(N \cap F\) is semi-closed set

**Example 1.7.**
Let \(X = \{0,1,2,3\}, \tau = \{\emptyset, \{0\}, \{1\}, \{0,1\}, X\}\),
For a topology defined on \(X\). let \(M = \{0\}\),
then \(\text{Int } \{0,2,3\} = \{0\} \subset \{0\} \subset \{0,2,3\}\). thus \(M\) is semi-closed set.

**Definition 1.8.** [6]
Let \((X, T)\) topological space. And \(M \subseteq X\), \(M\) is called \(alpha\-open\) set,
If \(M \subseteq \text{Int } \text{Cl } \text{Int } M\) . and we symbolize the family of \(alpha\-open\) sets with the symbol \(\alpha \ O (X)\).
Theorem 1.9. [7]
A topological space \((X,T)\), and \(M \subset X\), be \(\alpha\)-open set iff. There exists an open set \(N\), therefore \(N \subset M \subset \text{Int} \ Cl \ N\).

Proof:
If \(M \subset \text{Int} \ Cl \ M\), obviously \(\text{Int} \ M \subset M\).
So \(\text{Int} \ M \subset \text{Int} \ Cl \ M\).
Then \(\text{Int} \ M\) is an open set. If \(\text{Int} \ M = N\). \((N \text{ is open set})\)
But, \(M\) be an \(\alpha\)-open set then an open \(N\).
since \(N \subset M \subset \text{Int} N\). Therefore \(N \subset \text{Int} \ M \subset \text{Int} N\).
Hence \(\text{Int} Cl N \subset \text{Int} Cl \text{Int} M \subset \text{Int} \ Cl \text{Int} Cl N\).
Then \(M \subset \text{Int} Cl N \subset \text{Int} \ Cl \text{Int} M\),
Thus \(M \subset \text{Int} \ Cl \text{Int} M\).

Lemma 1.10. [8]
A topological space \((X,\tau)\), if \(M\) is open, \(M \subset X\). And \(N\) is open set,
Thus \(M \cap N\) is \(\alpha\)-open.

Proof:
Hence \(M\) is \(\alpha\)-open set, since \(H\) is open.
Then \(H \subset M \subset \text{Int} \ Cl \ H\), and the intersect them by the open set \(N\),
We get \(N \cap H \subset N \cap M \subset N \cap \text{Int} \ Cl \ H\),
Which implies \((N \cap H) \subset (N \cap M) \subset \text{Int} \ Cl (N \cap H)\).
And \((N \cap \text{Cl} \ H \subset \text{Cl} (N \cap H))\), take interior for both sides,
We have \(\text{Int}(N \cap \text{Cl} \ H) \subset \text{Int} \ Cl (N \cap H)\),
Such that \(\text{Int} N \cap \text{Int} Cl H \subset \text{Int} \ Cl N \cap \text{Int} Cl H\),
howerver \(\text{Int} N = N\). Thus \(N \cap \text{Int} Cl H \subset \text{Int} Cl (N \cap H)\).
Since the intersection of two open sets is open. the set \(M \cap N\) is \(\alpha\)-open set.

Example 1.11.
If \(X = \{1,2,3,4\}\), \(T_x = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, X\}\), a topology define on \(X\).
\(T_x^a = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}, X\}\), let \(M = \{1,2,4\}\), then \(M\) is \(\alpha\)-open
But \(M \notin T_x\).

2 On Semi \((\alpha,\alpha^*,\alpha^{**})\) Continuous Function

Definition 2.1. [1]
Given \(f : X \rightarrow Y\) be a function , so \(f\) is called semi \(\alpha\)-continuous, if and only if each open set \(N\) of \(Y\), then \(f^{-1}(N)\) a semi \(\alpha\)-open set of \(X\).

Remark 2.2. [1]
Each \(\alpha^*\)-continuous is \(\alpha\)-continuous, and semi \(\alpha\)-continuous. Then the opposite is not correct in general as in the following case. moreover, we have the following implication.

Example 2.3.
If \(X = \{1,2,3,4\}\), \(T_x = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, X\}\),
\(T_x^a = T_x \cup \{1,2,4\}\), and express \(f : X \rightarrow X\), also \(f(x_1) = 1, f(x_2) = f(x_3) = 3, f(x_4) = 4\).
It is simply shown that \(f\) is \(\alpha\)-continuous, then is not \(\alpha^*\)-continuous,
Hence \(f\) is \(\alpha\)-continuous, on the other hand it is not \(\alpha^*\)-continuous.

Example 2.4.
If \( X = \{1,5,9\} \), \( T_x = \{\emptyset, \{1\}, \{5\}, \{1,5\}, X\} \), \( T_x^\alpha = T_x \),

semi \( \alpha \) \( O(X) = T_x^\alpha \cup \{(5,9), \{1,9\}\} \). Let \( f: X \rightarrow X \), \( f(\chi_1) = 1 \),
\( f(\chi_2) = f(\chi_3) = 5 \). It is easily seen that \( f \) is semi \( \alpha \)-continuous, but \( f \) is not \( \alpha^* \)-continuous, thus \( \{5\} \in T_x^\alpha \), however \( f^{-1}(\{5\}) = \{5,9\} \notin T_x^\alpha \)

Therefore \( f \) is semi \( \alpha \)-continuous, then it is not \( \alpha^* \)-continuous.

**Remark: 2.5. [2]**
The ideas of \( \alpha^* \)-continuity and semi \( \alpha^* \)-continuity independent as the following example shows.

**Example 2.6.**
Let \( X = \{1,2,3,4\} \), \( T_x = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, X\} \), \( T_x^\alpha = T_x \cup \{(1,2,4)\} \).

semi \( \alpha \) \( O(X) = T_x^\alpha \cup \{(2,3,4), \{1,3,4\}, \{2,3\}, \{2,4\}, \{1,4\}, \{1,3\}\} \). If \( Y = \{5,7,9\} \), \( T_y = \{\emptyset, \{5\}, \{7\}, \{5,7\}, X\} \), \( T_y^\alpha = T_y \), semi \( \alpha \) \( O(Y) = T_y^\alpha \cup \{(7,9), \{5,9\}\} \), define to \( f: X \rightarrow Y \). By \( f(\chi_1) = f(\chi_4) = 7 \), \( f(\chi_3) = 9 \), \( f(\chi_2) = 5 \). So \( f \) is semi \( \alpha^* \)-continuous. But it is not \( \alpha^* \)-continuous.

Because \( \{7\} \in T_y^\alpha \), however \( f^{-1}(\{7\}) = \{1,4\} \notin \) Then \( f \) is semi \( \alpha^* \) – continuous.

Then it is not \( \alpha^* \) – continuous.

**Example 2.7.**
Let \( X = \{1,2,3,4\} \), \( T_x = \{\emptyset, \{1\}, \{2\}, \{1,2\}\{1,2,3\}, X\} \).

\( T_x^\alpha = T_x \cup \{(1,2,4)\} \),

Define Identity function.

Thus \( f(\chi_1) = f(\chi_2) = 2 \), \( f(\chi_3) = 4 \), \( f(\chi_4) = 3 \).

semi \( \alpha \) \( O(X) = T_x^\alpha \cup \{(2,3,4), \{1,3,4\}, \{2,3\}, \{2,4\}, \{1,4\}, \{1,3\}\} \).

Since \( f \) is \( \alpha^* \)-continuous. But it is not semi \( \alpha^* \)-continuous, because \( \{1,3\} \in \) semi \( \alpha \) \( O(X) \). However \( f^{-1}\{1,3\} = \{4\} \notin \) semi \( \alpha \) \( O(X) \).

As a result, \( f \) is \( \alpha^* \)-continuous, then \( f \) is not semi \( \alpha^* \)-continuous.

**Definition: 2.8. [1]**
suppose \( f: X \rightarrow Y \), then \( f \) is termed semi \( \alpha^* \)-continuous If and only if each \( N \) semi \( \alpha \)-open set of \( Y \). Thus \( f^{-1}(N) \) be a semi \( \alpha \)-open set of \( X \).

**Proposition: 2.9. [1]**
If \( f: X \rightarrow Y \) is \( \alpha^* \)-continuous, open and bijective then \( f \) is semi \( \alpha^* \)-continuous.

**Proof:**
If \( f: X \rightarrow Y \) be \( \alpha^* \)-continuous, open and bijective.

Let \( A \) a semi \( \alpha \)-open set of \( Y \). and there containing \( \alpha \)-open set, say \( N \) then \( N \subseteq A \subseteq Cl N \). So \( f^{-1}(N) \subseteq f^{-1}(A) \subseteq f^{-1}(Cl(N)) = Cl(f^{-1}(N)) \) \( \) as \( f \) is open \( \) . However \( f^{-1}(N) \in T_x^\alpha \), \( \) as \( f \) is \( \alpha^* \)-continuous, \( \) Therefore \( f^{-1}(N) \subseteq f^{-1}(A) \subseteq Cl(f^{-1}(N)) \)

Thus \( f^{-1}(A) \in \) semi \( \alpha \) \( O(X) \), so \( f \) is semi \( \alpha^* \)-continuous.

**Remark: 2.10.**
Let \( f: X \rightarrow Y \) and \( g: Y \rightarrow Z \) are two functions, thus \( f \) as well \( g \) exist \( \alpha \)-continuous, Thus \( fog: X \rightarrow Z \), we don’t need to prove \( \alpha \)-continuous as the example shows.

**Example 2.11.**
If \( X = \{4,5,6,7\} \), \( T_x = \{\emptyset, \{6\}, \{4,6\}, \{4,5,6\}, X\} \),

\( T_x^\alpha = T_x \cup \{(5,6), \{6,7\}, \{5,6,7\}, \{4,6,7\}\} \),
And \( Y = \{0,1,2\} \), \( T_y \) = \{0,2\}, \( Y \) = \{0,2\}, \( \{1,2\} \},

Define \( f: X \to Y \), \( f (x_1) = f (x_2) = 0, f (x_3) = f (x_4) = 1 \).

Also \( g: Y \to \mathbb{Z}, g (y_1) = g (y_3) = 6, g (y_2) = 4 \).

Then \( f \) and \( g \) are \( \alpha \)-continuous, but \( gof: X \to X \),

Where \( gof (x_1) = gof (x_2) = 6, gof (x_3) = gof (x_4) = 4 \).

Then \( gof \) is not \( \alpha \)-continuous, since \( \{6\} \) be an open set of \( X \). But \( (gof)^{-1} \{6\} \) = \{4,5\} be not \( \alpha \)-open set of \( X \). Therefore \( gof \) is not \( \alpha \)-continuous.

Definition. 2.12. \([1]\)

If \( f: X \to Y \) be a function, then \( f \) be called \( \text{semi } \alpha^{**} \)-continuous, iff for each \( N \) \( \text{semi } \alpha \)-open set in \( Y \), then \( f^{-1} (N) \) be open set in \( X \).

Theorem 2.13.

A function \( f: X \to Y \), then the following statements are equivalent,

I) \( f \) is \( \text{semi } \alpha^{**} \)-continuous.

II) \( f \) is \( \text{semi } \alpha^{**} \)-continuous at each point \( x \in X \). 

Proof:

(I) \( \Rightarrow \) (II)

If \( f: X \to Y \) is a \( \text{semi } \alpha^{**} \)-continuous.

And \( x \in X, M \) be open set of \( Y \) containing \( f(x) \).

Then \( x \in f^{-1} (M) \). Also, \( f \) is \( \text{semi } \alpha^{**} \)-continuous,

So \( N = f^{-1} (M) \) is \( \text{semi } \alpha \)-open set in \( X \) containing \( x \).

Therefore \( f(N) \subset M \).

(II) \( \Rightarrow \) (I)

If \( f: X \to Y \) is \( \text{semi } \alpha^{**} \)-continuous for all points in \( X \).

And \( M \) open set in \( Y \), let \( x \in f^{-1} (M) \),

Since \( M \) is open set in \( Y \) containing \( f(x) \),

By (II), at hand is \( \text{semi } \alpha \)-open set \( N \) of \( X \) containing \( x \).

Then \( f(x) \in f(N) \subset M \), therefore \( N \subset f^{-1}(M) \),

Hence \( f^{-1}(M) = \cup \{N; x \in f^{-1}(M)\} \), Thus \( f^{-1}(M) \) is \( \text{semi } \alpha \)-open.

Example 2.14.
Given \( X = \{3,5,7\} \), \( T_x = \{\emptyset ,\{3\},\{5\},\{3,5\},X\} \), \( T_x^\alpha = T_x \), \( \text{semi } \alpha \ O (X) = T_x^\alpha \cup \{\{3\},\{5\},\{3,5\}\} \).

If \( f \) is \text{identity-function}.

Via \( f (x_1) = 5, f (x_2) = f (x_3) = 3 \),

So \( f \) is \( \text{semi } \alpha \)-continuous , then \( f \) is not \( \alpha^* \)-continuous.

Since \( \{5\} \in T_x^\alpha \) and \( f^{-1} (\{5\}) = \{3\} \in T_x^\alpha \),

Therefore \( f \) is \( \text{semi } \alpha^{**} \)-continuous, however, \( f \) is not \( \alpha^* \)-continuous.

Lemma: 2.15.

If \( f: X \to Y \) is an \text{continuous and } open \), so \( f^{-1}(N) \in \alpha \ O(X) \), For every \( N \in \alpha \ O \ Y \).

Proof:

Given \( N \in \alpha \ O \ Y \), thus \( N \subset \text{Int } \text{Cl } \text{Int } (N) \), and \( f \) is \text{continuous},

We have \( f^{-1}(N) \subset f^{-1}(\text{Int } \text{Cl } \text{Int } (N)) \subset \text{Int } f^{-1}(\text{Cl } \text{Int } N) \).

Also by \( f \) open, \( f^{-1}(\text{Cl } \text{Int } N) \subset \text{Cl } f^{-1}(\text{Int } N) \), [ by if \( f: x \to Y \) is an open

Then \( f^{-1}(\text{Cl } A) \subset \text{Cl } f^{-1}(A) \), used for any \( A \subset Y \).

And since \( f \) is \text{continuous} , hence \( f^{-1}(\text{Int } N) \subset \text{Int } f^{-1} N \).
As a result \( f^{-1}(N) \subseteq \text{Int} \text{Cl} \text{Int} f^{-1}(N) \). Thus, \( f^{-1} N \in \alpha O(X) \). and the ease of finding it.

3 On a Homomorphism Function

**Definition 3.1.** [3]
If \( f: (X, \tau_X) \rightarrow (Y, \tau_Y) \) is a function, \( f \) is named **homeomorphism** if the injective, surjective, continuous and \( f^{-1} \) continuous.

**Remark 3.2.**
Clear that all homeomorphism function is always continuous. Then the opposite is not true as the example shows.

**Example 3.3.**
If \( f: (\mathbb{R}, \tau_{\mathbb{R}}) \rightarrow (\mathbb{R}, I); f(x) = x \), Identity function, for each \( x \in X \).
The function \( f \) is one-one, onto, continuous, but \( f^{-1} \) is not continuous.
So \( f \) is not homeomorphism.

**Theorem 3.4.** [3]
I- The bijective function \( f: (X, \tau_X) \rightarrow (Y, \tau_Y) \) is homeomorphism iff \( \text{Cl} (f^{-1}M) = f^{-1}(\text{Cl} M) \), s.t. \( M \subseteq Y \).
II- The bijective function \( f: (X, \tau_X) \rightarrow (Y, \tau_Y) \) is homeomorphism iff \( f^{-1}(\text{Int} M) = \text{Int}(f^{-1}M) \), such that \( M \subseteq Y \).

**Proof:**
(I)
Assume that \( \text{Cl} (f^{-1}M) = f^{-1}(\text{Cl} M) \), to show \( f \) is homeomorphism.
Since \( f \) is bijective, we must to prove \( f \) is continuous and \( f^{-1} \) is continuous. Thus \( \text{Cl} (f^{-1}M) = f^{-1}(\text{Cl} M) \).
Then \( \text{Cl}(f^{-1}M) \subseteq f^{-1}(\text{Cl} M) \), so \( f \) is continuous.
And \( f^{-1}(\text{Cl} M) \subseteq \text{Cl}(f^{-1}M) \), hence \( f^{-1} \) is continuous.
Similarly (by \( f \) is continuous iff \( f(\text{Cl} M) \subseteq \text{Cl}(f M), M \subseteq X \)), therefore \( f \) is continuous.
Conversely: assume that \( f \) is homeomorphism, to prove \( \text{Cl} (f^{-1}M) = f^{-1}(\text{Cl} M) \), since \( f \) is homeomorphism, then \( f, f^{-1} \) are continuous.
Thus \( \text{Cl}(f^{-1}M) \subseteq f^{-1}(\text{Cl} M) \) as well as \( f^{-1}(\text{Cl} M) \subseteq \text{Cl}(f^{-1}M) \)
As a result as \( \text{Cl} (f^{-1}M) = f^{-1}(\text{Cl} M) \).

(II):
assume that \( f^{-1}(\text{Int} M) = \text{Int}(f^{-1}M) \), to show \( f \) homeomorphism.
Since \( f \) is bijective, we need to verify \( f \) is continuous and \( f^{-1} \) is continuous.
So \( f^{-1}(\text{Int} M) = \text{Int}(f^{-1}M) \) then \( f^{-1}(\text{Int} M) \subseteq \text{Int}(f^{-1}M) \), thus \( f \) is continuous, by \( f \) is continuous iff \( f^{-1}(\text{Int} M) \subseteq \text{Int}(f^{-1}M), M \subseteq Y \).
And similarly \( f^{-1} \) is continuous, as result \( f \) is homeomorphism.
Conversely: assume that \( f \) is homeomorphism, show \( f^{-1}(\text{Int} M) = \text{Int}(f^{-1}M) \), so \( f \) and \( f^{-1} \) are continuous, then \( f^{-1}(\text{Int} M) = \text{Int}(f^{-1}M) \).

**Definition 3.5.** [3]
if known as two topological spaces \((X, \tau_x), (Y, \tau_y)\) are homeomorphic if there occurs a homeomorphism function from \((X, \tau_x) \to (Y, \tau_y)\) and symbolized by \((X, \tau_x) \cong (Y, \tau_y)\) or \((Y, \tau_y) \cong (X, \tau_x)\).

This means: \((X, \tau_x) \cong (Y, \tau_y)\) ifff occurs homeomorphism function \(f: (X, \tau_x) \to (Y, \tau_y)\).

**Remark 3.6.**

The relative \(\cong\) is the same relative on the family of topological space.

We must the relative \(\cong\) is reflexive, symmetric and transitive.

**References**


Baghdad University-Department of Mathematics (2020),76-82.


