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ON SOME: $(alpha, alpha\ star, alpha\ star\ star)$ $continuous\ function$

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ABSTRACT. In this paper a new class of functions, such that semi α–continuous function is introduced for topological spaces, and the second type is semi α^*–continuous function and three class is semi α^{**}–continuous function. We have taken in our study this continuous homeomorphism function (bijective = injective + surjective). we introduce using practical examples of mathematical formulas and considering them as a direct application to the validity of the observations. We also, study the relationship between these concepts that we referred to at the beginning of the research.

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Introduction

T.Noiri.[1] introduce the concept of new (α-$continuous$function), according to the source we got" On –continuous function". And when we return to the basic concepts, we studied with Y.Yousif, R. Mejed. [3]. And introduce semi-open by N. Levine,[4] "Semi-open sets and semi-continuity in Topological space", also N. Levine, [5] "Generalized Closed Sets in Topology Rend.". so we have an important concept to contest the results with the previous concepts O. Njastad, [6] "On Some Classes of Nearly Open Sets" *we* studied some characteristics of these functions by using definitions Also, we introduce using practical examples of mathematical formulas and considering them as a direct application to the validity of the observations. We also, study the relationship between these concepts that we referred to at the beginning of the resear1. The basic concepts

Definition 1.1. [4]
A topological space (X, τ), and $M \subseteq X$ be named ***semi*-open** set. exists open set $\boldsymbol{O}$, then $O \subset M \subset Cl\ O$.
Theorem 1.2. [4]

A topological space (X,τ), $M \subseteq X$, and M be $semi\text{-}open$ set if and only if $\boldsymbol{M \subset Cl\,Int\,M}$.
Proof :
If M is a $semi\text{-}open\ set$, and O is $open\ set$.
By defined $O \subset M \subset ClO$.
Then $Int\ M \subset M$. Hence $O \subset Int\ M \subset M \subset Cl\ O$,
Thus $Cl\ O \subset Cl\ Int\ M \subset Cl\ M \subset Cl\ O$, "$(Cl\ Cl\ O = Cl\ O)$"
So $M \subset Cl\ O$. Therefore $M \subset Cl\ Int\ M$.
On the other hand, M is a $semi\text{-}open\ set$. Hence $M \subset Cl\ Int\ M$.
Then $Int\ M \subset M$. So $Int\ M \subset M \subset Cl\ IntM$.
Therefore M it is $semi\text{-}open\ set$.

Example 1.3.
If $X = \{4,1,8\}$, $T = \{\emptyset, \{4\}, \{1\}, \{4,1\}, X\}$, a topology defined on X.
Thus $\{4,8\}, \{8,1\}$ are $semi\text{-}open$.
However $\{4,8\} \cap \{8,1\} = \{8\}$, it is not $semi$-open set.

Definition 1.4. [5]
a topological space (X,τ), too $M \subseteq X$, be called $\boldsymbol{semi\text{-}closed}$ set, then a $closed$ set C such that
$\boldsymbol{Int\ C \subset M \subset C}$.

Theorem 1.5. [5]
a topological space (X,τ), then $M \subseteq X$, M be $semi\text{-}closed$ set, iff $Int\ Cl\ M \subset M$.
Proof :
If M is a $semi$-closed set. Thus $M^C$is $semi\text{-}open\ set$.
So , $M^C \subset Cl\ Int\ M^C$.Then $Int\ Cl\ M \subset M$.
Conversely. so M is $semi\text{-}closed\ set$. Therefore $Int\ Cl\ M \subset M$.
Wherefore $M \subset Cl\ M$. Hence, $Int\ Cl\ M \subset M \subset Cl\ M$ and $Cl\ M$ is $closed\ set$.
As a result, M is $semi\text{-}closed\ set$
Corollary 1.6. [4]
The intersection of two $semi\text{-}closed\ sets$ is $semi\text{-}closed$ set in any $topological\ space$.
Proof :
Let N and F two semi-$closed\ sets$ in (X,τ) be a $topological\ space$. then by definition $semi\text{-}closed$ we get $Int\ Cl\ N \subseteq N$, and $Int\ Cl\ F$, therefore $Int\ Cl\ N \cap Int\ Cl\ F \subseteq N \cap F, but\ IntCl\ N \cap Int\ Cl\ F =$
$Int\ (ClN \cap Cl\ F)$, then $Int(Cl\ N \cap Cl\ F) \subseteq N \cap F$and
$Int(Cl(N \cap F) \subseteq Int(Cl\ N \cap Cl\ F)$, so $Cl(N \cap E) \subseteq ClN \cap Cl\ F$
Thus $Int\ \big(Cl(N \cap F)\big) \subseteq N \cap F$,
And from it, we find that $N \cap F$ is $semi\text{-}closed\ set$
Example 1.7.
Let $X = \{0,1,2,3\}$, $\tau = \{\emptyset, \{0\}, \{1\}, \{0,1\}, X\}$,
For a $topology$ defined on X. let $M = \{0\}$,
hence $Int\ \{0,2,3\} = \{0\} \subset \{0\} \subset \{0,2,3\}$. thus M is $semi\text{-}closed\ set$.

Definition. 1.8. [6]
Let (X,T) $topological\ space$. And $M \subseteq X$, M is called $\boldsymbol{\alpha\text{-}open}$ set,
If $\boldsymbol{M \subset Int\ Cl\ Int\ M}$. and we symbolize the family of $\alpha\text{-}open\ sets$ with the symbol $\boldsymbol{\alpha\ O}\ (\boldsymbol{X})$.

Theorem 1.9. [7]
A $topological\ space\ (X,T)$, and $M \subset X$, be α-$open\ set$ iff. There exists an open set N, therefore $N \subset M \subset Int\ Cl\ N$.

Proof :
If $M \subset Int\ Cl\ Int\ M$. obviously $Int\ M \subset M$.
So $Int\ M \subset M \subset Int\ Cl\ Int\ M$.
Then $Int\ M$ is an open set. If $Int\ M = N$. (N is $open\ set$)
But, M be an α-$open\ set$ then an $open\ N$.
since $N \subset M \subset IntN$. Therefore $N \subset Int\ M \subset M \subset Int\ N$.
Hence $Int\ Cl\ N \subset Int\ Cl\ IntM \subset Int\ Cl\ M \subset Int\ Cl\ Int\ Cl\ N$.
Then $M \subset Int\ Cl\ N \subset Int\ Cl\ Int\ M$,
Thus $M \subset Int\ Cl\ Int\ M$.

Lemma 1.10. [8]
A $topological\ space\ (X,\tau)$, if M is $open$, $M \subseteq X$. And N is $open\ set$,
Thus $M \cap N$ is α-$open$.

Proof :
Hence M is α-$open\ set$, since H is $open$.
Then $H \subset M \subset Int\ Cl\ H$, and the intersect them by the $open\ set\ N$,
We get $N \cap H \subset N \cap M \subset N \cap Int\ Cl\ H$,
Which implies($N \cap H) \subset (N \cap M) \subset Int\ Cl(N \cap H,)$.
And $(N \cap Cl\ H \subseteq Cl\ (N \cap H))$, take interior for both sides,
We have $Int(N \cap Cl\ H) \subseteq Int\ Cl\ (N \cap H)$,
Such that $IntN \cap Int\ Cl\ H \subseteq Int\ Cl\ N \cap Int\ Cl\ H$,
however $Int\ N = N$. Thus $N \cap Int\ Cl\ H \subseteq Int\ Cl\ (N \cap H)$.
Since the intersection of two $open\ sets\ is\ open$. the set $M \cap N$ is α-$open\ set$.

Example 1.11.
If $X = \{1,2,3,4\}$, $T_x = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, X\}$, a $topology$ define on X.
$T_x^\alpha = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}, X\}$, let $M = \{1,2,4\}$, then M is α-$open$
But $M \notin T_x$.

2 On $Semi\ (\alpha, \alpha^*, \alpha^{**})$ $Continuous$ Function

Definition 2.1. [1]
Given $f : X \to Y$ be a $function$, so f is called ***semi α -continuous,*** if and only if each ***open*** *set* N of **Y**, then $\boldsymbol{f^{-1}(N)}$ a ***semi α -open*** *set* of $\boldsymbol{X}$.

Remark 2.2. [1]
Each ***α*-continuous*** is ***α-continuous***, and ***semi α -continuous***. Then the opposite is not correct in general as in the following case. moreover, we have the following implication.

Example 2.3.
If $X = \{1,2,3,4\}$, $T_x = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, X\}$,
$T_x^\alpha = T_x \cup \{1,2,4\}$, and express $f : X \to X$, also $f\ (x_1) = 1, f\ (x_2) = f\ (x_3) = 3$, $f(x_4) = 4$.
It is simply shown that f is α-$continuous$, then is not α^*-$continuous$,
Hence f is α-$continuous$, on the other hand it is not α^*-$continuous$.

Example 2.4.

If $X = \{1,5,9\},\ T_x = \{\emptyset,\{1\},\{5\},\{1,5\},X\},\ T_x^{\alpha} = T_x,$
$semi\ \alpha\ O\ (X)$= $T_x^{\alpha} \cup \{\{5,9\},\{1,9\}\}$, Let $f: X \to X,\ \ f\ (x_1) = 1,$
$f\ (x_2) = f\ (x_3)\ = 5$. It is easily seen that f is $semi\ \alpha\text{-}continuous$, but f is not $\alpha^*\text{-}continuous$, thus $\{5\} \in T_x^{\alpha}$, however $f^{-1}(\{5\}) = \{5,9\} \notin T_x^{\alpha}$
Therefore f is $semi\ \alpha\text{-}continuous$, then it is not $\alpha^*\text{-}continuous$.

Remark: 2.5. [2]
The ideas of $\boldsymbol{\alpha^*\text{-}continuity}$ and $\boldsymbol{semi\ \alpha^*\text{-}continuity}$ independent as the following example shows.

Example 2.6.
let $X = \{1,2,3,4\},\ T_x = \{\emptyset,\{1\},\{2\},\{1,2\},\{1,2,3\},X\},\ \ T_x^{\alpha} = T_x \cup \{\{1,2,4\}\}$.
$semi\ \alpha\ O(X) =\ T_x^{\alpha} \cup \{\{2,3,4\},\{1,3,4\},\{2,3\},\{2,4\},\{1,4\},\{1,3\}\}$. If $Y = \{5,7,9\},\ T_y = \{\emptyset,\{5\},\{7\},\{5,7\},Y\},\ \ T_y^{\alpha} = T_y, semi\ \alpha\ O(Y) = T_y^{\alpha} \cup \{\{7,9\},\{5,9\}\}$, define to $f: X \to\ Y$. By $f(x_1) = f(x_4) = 7,\ f(x_3) = 9,\ \ f(x_2) = 5$. So f is $\boldsymbol{semi\ \alpha^*\text{-}continuous.}$ But it is **not** $\boldsymbol{\alpha^*\text{-}continuous}$,
Because $\{7\} \in T_y^{\alpha}$, however $f^{-1}\{7\} = \{1,4\}$ ∉Then f is $\boldsymbol{semi\ \alpha^*\ \text{–}\ continuous}$.
$Then\ it\ is\ \boldsymbol{not\ \alpha^* - continuous}$.

Example 2.7.
Let $X = \{1,2,3,4\ \},\ \ T_x = \{\emptyset,\{1\},\{2\},\{1,2\}\{1,2,3\},X\}$.
$T_x^{\alpha} = T_x \cup \{1,2,4\}$,
Define $\boldsymbol{Identity\ function}$.
Thus $f\ (x_1) = f\ (x_2) = 2,\ \ f\ (\ x_3) = 4,\ \ f\ (x_4\ ) = 3$.
$semi\ \alpha\ O$(X)= $T_x^{\alpha} \cup \{\{2,3,4\},\{1,3,4\},\{2,3\},\{2,4\},\{1,4\},\{1,3\}\}$.
Since f is $\alpha^*\text{-}\boldsymbol{continuous}$. But it is **not** $\boldsymbol{semi\ \alpha^*\text{-}continuous}$, because $\{1,3\} \in semi\ \alpha\ O(X)$. However $f^{-1}\{1,3\} = \{4\} \notin semi\ \alpha\ O(X)$.
As a result, f is $\boldsymbol{\alpha^*\text{-}continuous}$, then f is **not** $\boldsymbol{semi\ \alpha^*\text{-}continuous}$.

Definition: 2. 8. [1]
suppose $f: X \to Y$, then f is termed $\boldsymbol{semi\ \ \alpha^*\text{-}continuous}$ If and only if each N $\boldsymbol{semi\ \alpha\text{-}open}$ set of $\boldsymbol{Y}$. Thus $\boldsymbol{f^{-1}(N)}$ be a $\boldsymbol{semi\ \ \alpha\text{-}open}$ set of $\boldsymbol{X}$.

Proposition: 2.9. [1]
If $f: X \to Y$ is $\alpha^*\text{-}continuous$, open and bijective then f is $semi\ \alpha^*\text{-}continuous$.

Proof :
If $f: X \to Y$ be $\alpha^*\text{-}continuous, open\ and\ bijective$.
Let A a $semi\ \alpha\text{-}open\ set$ of Y. and there containing $\alpha\ \text{-}open\ set$, say N then $N \subseteq\ A \subseteq Cl\,N$. So $f^{-1}(N) \subseteq f^{-1}(A) \subseteq f^{-1}Cl(N) = Cl\left(f^{-1}(N)\right)$ **[as $\boldsymbol{f}$ is $\boldsymbol{open}$]**. However $f^{-1}(N) \in T_x^{\alpha}$, **[as $\boldsymbol{f}$ is $\boldsymbol{\alpha^*\text{-}continuous}$]**.Therefore $f^{-1}\ (N) \subseteq f^{-1}\ (A) \subseteq Cl\ (f^{-1}\ (N))$
Thus $f^{-1}A) \in semi\ \alpha\ O(X)$, so f is $semi\ \alpha^*\text{-}continuous$.

Remark: 2.10.
Let $f\ : X \to Y\ and\ \ g\ : Y \to Z\ are$ two functions, thus f as well g exist $\alpha\text{-}continuous$, Thus $fog: X \to Z$, we don't need to prove $\alpha\text{-}continuous$ as **the example shows.**

Example 2.11.
If $X = \{4,5,6,7\},\ T_x = \{\emptyset,\{6\},\{4,6\},\{4,5,6\},X\}$,
$T_x^{\alpha} = T_x \cup \{\{5,6\},\{6,7\},\{5,6,7\},\{4,6,7\}\}$,

And $Y = \{0,1,2\}$, $T_y = \{\emptyset, \{2\}, Y\}$, $T_y^{\alpha} = T_y \cup \{\{0,2\},\{1,2\}\}$,
Define $f: X \to Y$, $f(x_1) = f(x_2) = 0$, $f(x_3) = f(x_4) = 1$.
Also $g: Y \to Z$, $g(y_1) = g(y_3) = 6$, $g(y_2) = 4$.
Then f and g are α-*continuous*, but $gof: X \to X$,
Where $gof\,(x_1) = gof(x_2) = 6$, $gof(x_3) = gof(x_4) = 4$.
Then gof is not α-*continuous*, since $\{6\}$ be an open set of X. but $(gof)^{-1}\{6\} = \{4,5\}$ be not α-*open set* of X. Therefore gof is not α-continuous.

Definition. 2.12. [1]
If $f: X \to Y$ be a function, then f be called ***semi*** $\boldsymbol{\alpha}^{**}$-*continuous*, iff for each N ***semi*** $\boldsymbol{\alpha}$-*open set* in Y, then $\boldsymbol{f^{-1}(N)}$ be *open set* in X.

Theorem 2.13.
A function $f: X \to Y$, then the following statements are equivalent,

I) f is *semi* α^{**}-*continuous*.

II) f is *semi* α^{**}-*continuous* at each point $x \in X$,

Proof :

(I)→ (II)
If $f: X \to Y$ is a *semi* α^{**}-*continuous*.
And $x \in X$, M be *open set* of Y containing $f(x)$.
Then $x \in f^{-1}(M)$. Also, f is *semi* α^{**}-*continuous*,
So $N = f^{-1}(M)$ is *semi* α-*open set* in X containing x.
Therefore $f(N) \subset M$.
(II)→(I)
If $f: X \to Y$ is *semi* α^{**}-*continuous* for all points in X.
And M *open set* in Y, let $x \in f^{-1}(M)$,
Since M is *open set* in Y containing $f(x)$,
By (II), at hand is *semi* α-*open set* N of X containing x.
Then $f(x) \in f(N) \subseteq M$, therefore $N \subseteq f^{-1}(M)$,
Hence $f^{-1}(M) = \cup \{N; x \in f^{-1}(M)\}$, Thus $f^{-1}(M)$ is *semi* α-*open*.

Example 2.14.
Given $X = \{3,5,7\}$, $T_x = \{\emptyset, \{3\}, \{5\}, \{3,5\}, X\}$, $T_x^{\alpha} = T_x$, *semi* $\alpha\, O$ (X)= $T_x^{\alpha} \cup \{\{3,7\},\{5,7\}\}$,
If $\boldsymbol{f}$ is ***identity-function***.
Via $f(x_1) = 5$, $f(x_2) = f(x_3) = 3$,
So f is *semi* α-*continuous* , then f is not α^*-*continuous*
Since $\{5\} \in \tau_x^{\alpha}$, and $f^{-1}(\{5\}) = \{3\} \in T_x^{\alpha}$,
Therefore f is *semi* α^{**}-*continuous*, however, f is not α^*-*continuous*.

Lemma: 2.15.
If $f: X \to Y$ is an *continuous* and *open* , so $f^{-1}(N) \in \alpha\, O(X)$, For every $N \in \alpha\, O\,(Y)$.
Proof:
Given $N \in \alpha\, O\,(Y)$, thus $N \subseteq Int\, Cl\, Int\,(N)$, and f is *continuous*,
We have $f^{-1}(N) \subseteq f^{-1}(Int\, Cl\, Int\, N) \subseteq Int\, f^{-1}(Cl\, Int\, N)$,
Also by f open, $f^{-1}(Cl\, Int\, N) \subseteq Cl\, f^{-1}(Int\, N)$, [by if $f: x \to Y$ is an open
Then $f^{-1}(Cl\, A) \subseteq Cl\, f^{-1}$ A , used for any $A \subseteq Y$.
And since f is *continuous* , hence $f^{-1}(Int\, N) \subseteq Int\, f^{-1}\, N$.

As a result $f^{-1}(N) \subseteq Int\ Cl\ Int\ f^{-1}(N)$, Thus, $f^{-1}\ N \in \alpha\ O\ (X)$. **and the ease of finding it.**

3 On a Homomorphism Function

Definition 3.1. [3]
If $f: (X, \tau_x) \to (Y, \tau_y)$ is a function, f is named $\boldsymbol{homeomorphism}$ if the injective, surjective, continuous and f^{-1} continuous.

Remark 3.2.
Clear that all $homeomorphism$ function is always $continuous$. Then the opposite is not true as the example shows.
Example 3.3.
If $f: (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$; $f(x) = x$, $Identity\ function$, for each $x \in X$.
The function f is one-one, onto, $continuous$, but f^{-1} is not $continuous$.
So f is not $homeomorphism$.

Theorem 3.4. [3]

I- The bijective $function\ f: (X, \tau_x) \to \left(Y, \tau_y\right)$ is $homeomorphism$ iff $Cl\ (f^{-1}M) = f^{-1}(Cl\ M)$, s.t. $M \subseteq Y$.

II- The $bijective\ function\ f: (X, \tau_x) \to \left(Y, \tau_y\right)$ is $homeomorphism$ iff $f^{-1}(Int\ M) = Int(f^{-1}M)$, such that $M \subseteq Y$.

Proof :
(I)
Assume that $Cl\ (f^{-1}M) = f^{-1}(ClM)$, to show f is $homeomorphism$,
Since f is $bijective$, we must to prove f is $continuous$ and f^{-1} is $continuous$. Thus $Cl(\ f^{-1}M) = f^{-1}(Cl\ M)$,
Then $Cl(f^{-1}\ M) \subseteq f^{-1}(Cl\ M)$, so f is $continuous$.
And $f^{-1}(ClM) \subseteq Cl(f^{-1}M)$, hence $f^{-1}$is $continuous$. similarly
(by f is $continuous$ iff $f(Cl\ M) \subseteq Cl(f\ M)$, $M \subseteq X$), therefore f is c$ontinuous$.
Conversely: assume that f is $homeomorphism$, to prove $Cl\left(f^{-1}(M)\right) = f^{-1}\ (Cl\ M)$, since $f\ is\ homeomorphism$, then f, f^{-1} are $continuous$,
Thus $Cl(f^{-1}\ M) \subseteq f^{-1}(Cl\ M)$ as well as $f^{-1}(Cl\ M) \subseteq Cl(f^{-1}\ M)$
As a result as $Cl\ (f^{-1}M) = f^{-1}\ (Cl\ M)$.
(II) :
assume that $f^{-1}(Int\ M) = Int(f^{-1}\ M)$, to show $f\ homeomorphism$.
Since f is bijective, we need to verify $f$is $continuous$ and $f^{-1}$is $continuous$.
So $f^{-1}\ (IntM) = Int(f^{-1}M)$ then $f^{-1}(IntM) \subseteq Int(f^{-1}M)$, thus f is $continuous$, by [$\boldsymbol{f}$ **is** $\boldsymbol{continuous}$ **iff** $\boldsymbol{f^{-1}(IntM) \subseteq Int\ (f^{-1}M), M \subseteq Y}$]
And similarly f^{-1} is $continuous$, as result f is $homeomorphism$.
Conversely: assume that f is $homeomorphism$, show $f^{-1}(IntM) = Int(f^{-1}M)$, so $f$and $f^{-1}$are continuous, then $f^{-1}(IntM) = Int(f^{-1}M)$.
Definition 3.5. [3]

if known as two $topological\ spaces$ $(X,\tau_x),(Y,\tau_y)$ are $h\boldsymbol{omeomorphic}$ if there occurs a $homeomorphism$ function from $(X,\tau_x) \to (Y,\tau_y)$ and symbolized by$(X,\tau_x) \cong (Y,\tau_y)$ or $(Y,\tau_y) \cong (X,\tau_x)$.

This means: $(\boldsymbol{X},\boldsymbol{\tau_x}) \cong (\boldsymbol{Y},\boldsymbol{\tau_y})$ **iff occurs** $\boldsymbol{homeomorphism\ function\ f}: (\boldsymbol{X},\boldsymbol{\tau_x}) \to (\boldsymbol{Y},\boldsymbol{\tau_y})$.

Remark 3.6.

The relative $\cong$ is the same relative on the family of $topological\ space$.

We must the relative $\cong$ is $\boldsymbol{reflexive, symmetric\ and\ transitive}$.

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