Shuwaie, J., & Hussain, A. (2022). On Feebly Open Set and Its Properties. *Akkad Journal of Multidisciplinary Studies*, *2*(1), 1-11.

ON FEEBLY OPEN SET AND ITS PROPERTIES

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Received: December 2020
1st Revision: March 2021
Accepted: June 2021

***JEL Classification**:* example
D02, O17, P31

ABSTRACT. This work aims to present new types of sets named feebly open sets that were studied and identified some of their properties and found relationships with other sets where we obtained some results that show the relationship between sets that were obtained using the set from types (f-open)

***Keywords** feebly-open, feebly-closed, feebly-closure.*

Introduction

One of a topology's most important and fascinating concepts is the idea of feebly separation characteristics. In 1963, N.levin[1] proposed the concept of a semi-open set. S.N Maheshwari and R. Prasad[2], used a semi-open set to characterize and investigate novel divisions known as semi-detachment aphorisms. Maheshwari S. N and Tapiu[3] initiated the study of feebly open in 1978. In 2019[4], Ail Khalaf Hussain Al-Hachami presented the idea of some feeble separation properties, it is demonstrated that every feebly-T_1 is semi-T_1 and every feebly-$T_{1/2}$ is feeble-T_0. Aaad Aziz Hussan Abdulla in [5] presented the idea of a semi-feebly open (if-open) set. In 2021, Ail.Al Kazaragy, Faik. Mayah and Ail Khala Hussain Al-Hachami [6]. introduced defined semi-θ -axioms. Zainab Awad Kadhum and Ail Khala Hussain[7] defied $ii\ \delta_g$ –closed set in topological spaces, it is demonstrated that each ii-$T_{3/4}$ space is ii-$T_{1/2}$ space "The goal of this study is to provide some characterizations of some feebly open properties".

1. Preliminaries

Definition 1.1: [1]

Let (X, τ) be a topological space. A subset A of X is called to be semi-open (s-open) set if there exists an open set U such that $U \subset A \subset \overline{U}$. Or equivalent[1], A is called s-open in $X \Leftrightarrow A \subset \overline{(A^o)}$.

Definition 1.2: [8]

Let (X, τ) be a topological space. A subset A of X is called to be semi-closed (s-closed) set if there exists a closed set U such that $U^o \subset A \subset U$. Or equivalent[9], A is called s-closed in $X \Leftrightarrow \left(\overline{A}\right)^o \subset A$.

Definition 1.3:[10].

If B is a subset (X, τ), then semi-closure of B, denoted by $\overline{B}^s$ is the the intersection of all semi-closed subsets of X containing B and semi-interior of B, denoted by B^{os} is the union of all semi-open subsets of X containing B.

Proposition 1.4:[11]

Let (X, τ) be a topological space. If $A \subseteq X$ then $\overline{A}^s = A \cup \left(\overline{A}\right)^o$.

Note 1.5:[11]

If B_1 and B_2 are semi-open in a topological space (X, τ) then $B_1 \cap B_2$ is not necessarily a semi-open set.

To see this, consider the following counter-example:

Example 1.6:

Let $X = \{1, 2, 3, 4\}$, $\tau = \{ \emptyset, \{3\}, \{2\}, \{2, 3\}, \{1, 2, 3\}, X\}$ be a topology defined on X. Let $B_1 = \{2, 4\}$ and $B_2 = \{1, 3, 4\}$ are semi-open set.

But $\{2, 4\} \cap \{1, 3, 4\} = \{4\}$ is not semi-open set.

Theorem 1.7:[1]

Let X, Y be topological spaces and $X \times Y$ be the product topology. If A is f-open set in X and B is f-open set in Y, then $A \times B$ is f-open set in $X \times Y$.

Definition 1.8:[12]

A subset B of a topological space (X, τ) is called an α-open set

if $B \subset \left(\overline{(B^o)}\right)^o$.

Definition 1.9:[12]

A subset B of a topological space (X, τ) is called an α-closed set

If $\overline{((\bar{B})^o)} \subset B$.

2. The Main Results

Definition 2.1:[13]

A subset A of a topological space(X, τ) is called feebly open (f$-$open) set if there exists an open set O such that $O \subseteq A \subseteq \overline{O}^s$.

Theorem 2.2:

Let (X, τ) be a topological space then A is called an f-open set if and only if $A \subseteq \overline{(A^o)}^s$.

Proof.

Let A is f-*open set* , we must proof $A \subseteq \overline{(A^o)}^s$.

Since A is an f-open set, then there is an open set O such that $O \subseteq A \subseteq \overline{O}^s$ [Definition (2.1)].

Since $A^o \subseteq A$, consequently $O \subseteq A^o \subseteq A \subseteq \overline{O}^s$.

Hence $\overline{O}^s \subseteq \overline{(A^o)}^s \subseteq \overline{A}^s \subseteq \overline{\left(\overline{O}^s\right)}^s = \overline{O}^s$

Since $A \subseteq \overline{O}^s$ and $\overline{O}^s \subseteq \overline{(A^o)}^s$ implies $A \subseteq \overline{(A^o)}^s$.

To prove the other direction

Let $A \subseteq \overline{(A^o)}^s$, we must prove A to be an f-open.

Since $A^o \subseteq A$, implies $A^o \subseteq A \subseteq \overline{(A^o)}^s$ [Definition(2.1)].

Then A is f-open.

Proposition 2.3:

Every $open$ set is f-$open$ set.

Proof.

Let A be $open$ set $then$ $A = A^o$.

Since $A \subseteq \overline{A}^s$, then $A \subseteq \overline{(A^o)}^s$

observe that A is an f-open set.

Remark 2.4:

The converse of [Proposition (2.3)] is not necessarily true as shown by

The following example.

Example 2.5:

Let X = {1, 2, 3}, τ = {X, ∅, {1}}be a topology defined on X.

Let A = {1, 2}, then A is f-open set

Since $\overline{\{1\}}^s = X$ and $\{1\} \subseteq \{1,2\} \subseteq X$.

Hence A is an f-open set but not an open set.

Proposition 2.6:

If A is the f-open set and B is the f-open set, then $A \cap B$ is not necessarily the f-open set.

As shown by the following example.

Example 2.7:

Let X = {1, 2, 3, 4, 5}, τ = {X, {1},{4},{1,4},{3,4},{1,3,4},{1,2,4,5}, ∅ }be

a topology defined on X.

s-open sets ={∅, X, {1}, {4}, {1, 4}, {3, 4}, {1, 3, 4}, {1, 2, 4, 5}, {1, 2}, {1, 3}, {1, 2, 3}, {2, 3, 4}, {3, 4, 5}, {1, 2, 3, 4},{1, 2, 4, 5},{1,3,4,5},

{2, 3, 4, 5}, {1, 4},{1 ,2 , 4}, {1, 3, 4},{1, 4, 3}.

s-closed sets = {$\emptyset$, X, {2, 3, 4, 5},{1 ,2, 3, 5},{3, 4, 5}, {2, 3, 4},{1, 2, 3},

{3, 4}, {2, 5}, {1, 5},{1,2},{5},{2},{3},{1},{2,3,5},{3,5},{2,3}.

f-open sets = {$\emptyset$, X, {1}, {4}, {1, 4},{3, 4}, {1, 2, 4},{1 ,3 , 4},{1,2,3,4},

{1, 3, 4, 5}, {1, 2, 3, 5}.

It is clear {1, 3, 4, 5}, and {1, 2 ,3 , 5} is f-open set but

{1, 3, 4, 5} ∩ {1, 2, 3, 5} = {1, 3, 5} is not f-open set.

Proposition 2.8:

Each f-open set is a semi-open set.

Proof.

Let A be an f-open set, then there exists an open set such that

$U \subseteq A \subseteq \overline{U}^{s}$ [Definition (2.1)].

Since $\overline{U}^{s} \subseteq \overline{U}$ implies $U \subseteq A \subseteq \overline{U}^{s} \subseteq \overline{U}$

Then $U \subseteq A \subseteq \overline{U}$

Thus A is a semi-open set.

Remark 2.9:

The converse of [Proposition (2.8**)]** is not necessarily true as shown by

The following example.

Example 2.10:

Let X = {1,2,3}, τ = {X, $\emptyset$, {1}, {3}, {1, 3}} be a topology defined on X.

s-open sets = {X, $\emptyset$, {1}, {3}, {1, 3}, {1, 2}, {2, 3}}.

s-closed sets = $\{X, \emptyset, \{2, 3\}, \{1, 2\}, \{2\}, \{1\}, \{3\}\}$.

f-open sets = $\{X, \emptyset, \{1\}, \{3\}, \{1, 3\}\}$.

It is clear {1, 2} is semi-open but f-open set.

<u>Theorem 2.11:[13]</u>

Let (X, τ) be a topological space, then A is an f-open set if and only if

$A \in t^{\alpha}$.

Proof.

If A is an f-open set, then there exists an open set U such that

$U \subseteq A \subseteq \overline{U}^{s}$.

We have $\overline{U}^{s} = U \cup \left(\overline{U}\right)^{o} = \left(\overline{U}\right)^{o}$ [Proposition (1.4)]. Then $U \subseteq A \subseteq \left(\overline{U}\right)^{o}$.

Since $\left(\overline{(A^{o})}\right)^{o}$=$\left(\overline{U}\right)^{o}$.

Then $A \subseteq \left(\overline{(A^{o})}\right)^{o}$

Thus A is α-open set.

To prove the other direction

If $A \in t^{\alpha}$, we have $A \subseteq \left(\overline{(A^{o})}\right)^{o}$.

Since $A^{o} \subseteq A$, consequently $A^{o} \subseteq A \subseteq \left(\overline{(A^{o})}\right)^{o}$, so if A^{o}=U,

Then $U \subseteq A$, Since $\overline{U}^{s} \subseteq \overline{U}^{o}$[Proposition (1.4)].

Hence $U \subseteq A \subseteq \overline{U}^{s}$

Thus A is an f-open set.

<u>Definition 2.12:[13]</u>

A subset A of a topological space (X, τ) is called f-closed if the complement is f-open set.

Lemma 2.13:[14]

A subset A of a topological space (X, τ). then

$\left(\overline{A}^{s}\right)^{c} = (A^{c})^{os}$.

Theorem 2.14:

Let (X, τ) be a topological space .then A is f-closed if and only if

$\left(\overline{A}\right)^{os} \subseteq A$.

Proof.

Let A is f-closed, we must proof that $\left(\overline{A}\right)^{os} \subseteq A$.

Since A is f-closed, then A^{c} is f-open.

Therefore, $A^{c} \subseteq \overline{((A^{c})^{o})}^{s}$ [Theorem (2.2)].

Then $\left(\overline{((A^{c})^{o})}^{s}\right)^{c} \subseteq A$

Thus $\left(\overline{A}\right)^{os} \subseteq A$ [Lemma (2.13)].

Conversely, let $\left(\overline{A}\right)^{os} \subseteq A$, we must proof A is f-closed.

Then $A^{c} \subseteq \left(\left(\overline{A}\right)^{os}\right)^{c}$ implies $A^{c} \subseteq \overline{((A^{c})^{o})}^{s}$ [Lemma (2.13)].

Then A^{c} is f-open set [Theorem (2.2)].

Thus A is f-closed.

Proposition 2.15:

If A is a closed set then A is an f-closed set.

Proof.

Let A is closed set then $A = \overline{A}$.

Since $A^{os} \subseteq A$ then $\left(\overline{A}\right)^{os} \subseteq A$.

Hence A is f-closed.

Remark 2.16:

The converse of [Proposition (2.15)] is not necessarily true as shown by the following example.

Example 2.17:

Let X = {1, 2, 3}, τ = { X, ∅, {1}} be a topology defined on X.

Let A = {3}.

It is clear A is an f-closed set but not a closed set.

Corollary 2.18:

If A_λ is a family of f-closed sets, then $\cap A_\lambda$ is an f-closed set.

Corollary 2.19:

If A is an f-open set and B is an f-closed set, then $A \cup B$ is not necessarily f-open set. As shown by the following example

Example 2.20:

In example (1.3.10), A = {3} and B = {2, 5} are f-closed sets but $A \cup B$ = {2, 3, 5} is not f-closed set.

Proposition 2.21:

If A is f-open set and U open set in X, then $A \cap U$ is f-open in X and U.

Proof.

We proof that $A \cap U \subseteq \overline{((A \cap U)^o)}^s$.

Since $\overline{((A \cap U)^o)}^s = \overline{(A^o \cap U^o)}^s \subseteq \overline{(A^o)}^s \cap \overline{(U^o)}^s$.

Then $\overline{((A \cap U)^o)}^s = \overline{(A^o)}^s \cap \overline{(U^o)}^s$.

Let $x \in A \cap U$ then $x \in A$ and $x \in U$.

Since A is an f-open set then $A \subseteq \overline{(A^o)}^s$ [Theorem (2.2)].

Therefore, $x \in \overline{(A^o)}^s$. Since $U \subseteq \overline{U}^s$, then $x \in \overline{U}^s$.

Hence $x \in \overline{(A^o)}^s \cap \overline{U}^s$.

Then $A \cap U \subseteq \overline{((A \cap U)^o)}^s$.

Proposition 2.22:

If A is an f-closed set then A is a semi-closed set.

Proof.

Let A is f-closed set then A^c is f-open set

Hence A^c is semi-open

Thus A is a semi-closed set.

Remark 2.23:

The converse of [Proposition (2.22)] is not necessarily true as

in example (2.10), A = {3} is semi-closed set but not f-closed set.

Definition 2.24:

The letter $\overline{A}^f$ denotes the feeble closure of A, which is the intersection of all

feebly closed sets containing A.

Evidently, $\overline{A}^f = \cap\{W \subseteq X;$ feebly closed : $A \subseteq W\}$

Proposition 2.25:

Let X be a topological space and $A, B \subseteq X$, then

1. A is an f-closed set if and only if $A = \overline{A}^{f}$.

2. $\overline{A}^{f} \subseteq \overline{A}$.

3. $\overline{A}^{f} = \overline{\left(\overline{A}^{f}\right)}^{f}$.

4. If $A \subseteq B$ then $\overline{A}^{f} \subseteq \overline{B}^{f}$.

Proof.

1- (⇒) Let A is a f-closed set. Since $A \subseteq \overline{A}^{f}$. Then $\overline{A}^{f} \subseteq A$ (since $\overline{A}^{f}$ is the smallest f-closed set containing A), then $A = \overline{A}^{f}$.

(⇐) Let $A = \overline{A}^{f}$. Then $\overline{A}^{f}$ is an f-closed set. as $A = \overline{A}^{f} \Rightarrow A$ is an f-closed set.

2- Let $x \in \overline{A}^{f}$ and A is an f-closed set, $A = \overline{A}^{f} \Rightarrow x \in A \subseteq \overline{A}$. Then $x \in \overline{A}$.

Therefore, $\overline{A}^{f} \subseteq \overline{A}$.

3- Since $\overline{A}^{f}$ is f-closed set, then $\overline{A}^{f} = \overline{\left(\overline{A}^{f}\right)}^{f}$. by(2).

4- Let $A \subseteq B$ and $B \subseteq \overline{B}^{f}$, then $A \subseteq \overline{B}^{f} \Rightarrow \overline{B}^{f}$ is an f-closed set containing A.

Since $\overline{A}^{f}$ is the smallest f-closed set containing A. Then $\overline{A}^{f} \subseteq \overline{B}^{f}$.

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Conclusion

In this paper it was concluded that there is a relationship between a known open set and a feebly open set there is also a relationship between a feebly open set and a semi-open set and a relationship between a closed set and a feebly closed set.

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