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Review of solution Fractal Differential Equations via the Riemann-Liouville

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ABSTRACT. In this paper, the definitions, concepts, and basics of linear and nonlinear fractal differential equations of both homogeneous and heterogeneous types were studied, and some methods for solving fractal differential equations were presented with an explanation of each method and how to solve the equation in it An example was taken for each method that was presented to show how to solve in this way. Among the methods that were presented are (the special method, the Laplace transform Method, the Inverse Fractional Shehu Transform Method, The Power Series Method, The generalized Mittag-Leffler method, Spline Interpolation Techniques for Approximating the Solution of Fractional Differential Equations.

Keywords: Riemann –Liouville, Fractal Differential Equations, Power Series Method

Introduction

Fractional calculus owes its origin to a question of whether the meaning of a derivative to an integer order n could be extended to still be valid when is not an integer. This question was first raised by L'Hopital on September 30th, 1695. On that day, in a letter to Leibniz, he posed a question about $\frac{D^n x}{Dx^n}$ Leibniz's notation for the n^{th} derivative of the linear function $f(x) = x$. L'Hopital curiously asked what the result would be if $n = \frac{1}{2}$ Leibniz responded that it would be "an apparent paradox, from which one-day useful consequences will be drawn"[2]. Following this unprecedented discussion, the subject of fractional calculus caught the attention of other great mathematicians, many of whom directly or indirectly contributed to its development. They included Euler, Laplace, Fourier, Lacroix, Abel, Riemann, and Liouville In 1819, Lacroix became the first mathematician to publish a paper that mentioned a fractional derivative [34]. Commonly these fractional integrals and derivatives were not known to many scientists and up until recent years, they have been only used in a purely mathematical context, but during these last few decades these integrals and derivatives have been applied in many science contexts due to their frequent appearance in various applications in the fields of fluid mechanics, viscoelasticity, biology, physics, image processing, entropy theory, and engineering [5,6,7,8,]. It is well known that the fractional order differential and integral operators are non-local operators. This is one reason why fractional calculus theory provides an excellent instrument for the description of memory and hereditary properties of various physical processes. For example, half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models [1,2,3,4]. There are also many mathematical methods for obtaining solutions to fractional differential equations, such as the field analysis method [26] variable frequency method [27], and the new iteration method [28] differential transformation method. [29] homotopy analysis method, homotopy perturbation method, and reductive partial differential transformation method. [30] Fractional residual power series method [31]In this research Definitions and basic concepts of linear and nonlinear fractal differential equations of both homogeneous and inhomogeneous types were presented, and some methods of solving fractal differential equations were presented with an explanation of each method and how to solve the equation in it. An example was taken for each method presented to show how to solve in this way, and the examples were compared to show the strengths and weaknesses of each method.

1. Basic Concepts of Fractal Differential Equations to Riemann-Liouville

Definition(1) In the theory of fractional calculus, the entire gamma function Γ(t) plays a substantial turn. An inclusive definition of $\Gamma(t)$ is that supplied by the Euler limit [40]

$$\Gamma(t) = \lim_{N\to\infty} \frac{N!Nt}{t(t+1)(t+2)\dots(t+N)} \quad , t > 0 \qquad (1)$$

The useful integrated conversion formula is specified by:

$$\Gamma(z)=\int_0^\infty t^{z-1}\, e^{-t}\, dt \qquad ,R(t) > 0 \qquad (2)$$

1.1 . Riemann-Liouville Fractional Calculus

This section presents some definitions, of fractional differential equations with the Riemann-Liouville definitions. The French mathematician, Lacroix [Structures, 1819] write on differential and integral calculus in which he devoted less than two pages to this topic starting with $y = x^m$, m a positive integer, he found the nth derivative to be[41]

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n} \qquad (3)$$

by using Legendre's symbol (Γ) which denotes the generalized factorial, eq (3) becomes

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \,, m > n \qquad (4)$$

1.2. Liouville's Definitions for a Fractional Derivative

Liouville divided its definition into two parts as Liouville's [41] first and second definitions. Liouville's starting point is the known result for derivatives of integral order

$$D^n\, e^{at} = a^n e^{at} \qquad (5)$$

which he extended naturally to derivatives of arbitrary order

$$D^v\, e^{at} = a^v e^{at} \qquad (6)$$

He expanded the function $y(t)$ in the series

$$y(t) = \sum_{t=0}^{\infty} c_n\, e^{\,ant} \qquad (7)$$

• Liouville's First Definition of a Fractional Derivative :
He assumed that the derivative of arbitrary order y(t) is

$$D^\alpha\, y(t) = \sum_{t=0}^{\infty} c_n\, a_n^v\, e^{\,ant} \qquad Re(an) > 0 \qquad (8)$$

• Liouvlille's Second Definition for a Fractional Derivative is[41][35]

$$D^V x^{-a} = (-1)^v \frac{\Gamma(a+v)}{\Gamma(a)} t^{-a-v} \tag{9}$$

where a > 0 and $t > 0$. But Liouville's definitions were too narrow to last, the first definition it restricted to the function of the form in equation (9), the second definition is useful only for functions of the type, and the second definition is useful only for functions of the type $\boldsymbol{t^{-a}}$ Riemann was influenced by one of Liouville's memoirs in which Liouville wrote, the ordinary differential equation $\frac{d^n y}{dx^n} = 0$ has the complementary solution $y_c = c_\circ + c_1 t + c_2 t^2 + \cdots c_{n-1} t^{n-1}$ (10)

Believe Riemann saw fit to add a complementary function to his definition of a fractional integration

1.3. Riemann's Definition of Fractional Integral

The Riemann fractional integral [36] is defined as

$$cD_t^{-v} y(t) = \frac{1}{\Gamma(v)} \int_c^t (t-x)^{v-1} y(x)dx + \Psi(t) \tag{11}$$

where Ψ(x) is called a complementary function. Riemann defined $\Psi(t)$ because of the ambiguity of the lower limit of integration c. when $t > c$ we have Riemann's definition but without a complementary function

$$cD_t^{-v} y(t) = \frac{1}{\Gamma(v)} \int_c^t (t-x)^{v-1} y(x)dx + \Psi(t) \qquad Re(v) > 0 \tag{12}$$

when c = −∞ equation (1.7) is equivalent to Liouville's definitions.

The $\alpha\, th$ order left and right Riemann –liouville integrals of function $y(t)$ are defined on the interval (a,b) as following[1][18][22]

$$aI_t^\alpha\ y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{y(s)}{(t-s)^{1-\alpha}}\, ds \tag{13}$$

$$tI_b^\alpha\ y(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{y(s)}{(t-s)^{1-\alpha}}\, ds \tag{14}$$

Where $\alpha > 0$ are called fractional integrals of the order $\alpha,$ they are sometimes called left-sides and right–sided fractional integrals respectively.

1.4 . Types of Fractal Differential Equation

The ordinary differential equations, which contain two types of equations, which are linear differential equations, both homogeneous and non-homogeneous, and non-linear differential equations, as well as both homogeneous and inhomogeneous types, here also fractal differential equations are divided into two types, as follows

1.4.1. Linear Fractal Differential Equation

Definition(2) The fractal differential equation is called by the formula

$$(D^{\alpha n} + a_{n-1} D^{\alpha n-1} + \cdots + a_1 D^{\alpha 1} + a_\circ) y(t) = f(x) \tag{15}$$

With $\alpha \in R$ the condition $y^k(0) = 0$ $\qquad k = 0,1,2, \ldots, n-1$

Is linear fractal differential equation non-homogeneous, but if the $f(x) = 0$ then it is homogeneous[32].

1.4.2. Non–linear fractal differential equation

Definition (3) Alinear FDE is an equation of from
$(D^{\alpha n} + a_{n-1} D^{\alpha n-1} + \cdots + a_1 D^{\alpha 1} + a_\circ)y(t) = f(x)$ (16)
(1.16)
With $\alpha \in R$ the condition $y^k(o) = b_k$ $k = 0,1,2,\ldots,n-1$
And D^α is defined as the Riemann –Liouville derivative of function $y(t)$,
An equation that is not linear is called non –linear fractal differential equation[32].

1.5. The composition of Riemann-Liouville fractional integral and derivative

1. $I^\alpha\left(I^\beta y(t)\right) = I^\beta\left(I^\alpha y(t)\right) = I^{\alpha+\beta} y(t)$ [41],[42] *where* $\alpha > 0$ *and* $\beta > 0$ (17)
2. $D^\alpha\left(D^\beta y(t)\right) = D^\beta\left(D^\alpha y(t)\right) = D^{\alpha+\beta} y(t)$ [41], [42] (18)
3 . $D^\alpha(I^\alpha y(t)) = y(t)$ [41] where $\alpha > 0, t > 0$ (19)
4. $D^\alpha\left(\lambda y(t) + \mu g(t)\right) = \lambda D^\alpha y(t) + \mu D^\alpha g(t)$ [41] (20)
5. $D^\alpha\left(k y(t)\right) = k D^\alpha y(t)$ $\alpha > 0$ [42] (21)
6. $D^\alpha\left(y(t).g(t)\right) = [D^\alpha y(t)].g(t) + y(t)[D^\alpha g(t)]$ [41] (22)
7. $D^\alpha\left(y(t) + g(t)\right) = D^\alpha y(t) + D^\alpha g(t)$ $\alpha \in R$ [41] (23)
Lemma(1):

Let α > 0, then the differential equation[9]

$D^\alpha y(t) = 0$ (24)

It has solution

$y(t) = c_\circ + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ $, c_i \in R, i = 0,1,2,\ldots.n-1$ (25)

***1.6 The Derivative of Fractional Differential Equation**[41],[45]*

1 . $D^\alpha(t^n) = \frac{\Gamma(n-1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}$ (26)
2 . $D^\alpha(\sin \alpha t) = a^\alpha \sin\left(at + \frac{\pi}{2}\alpha\right)$ (28)
3 . $D^\alpha(\cos \alpha t) = a^\alpha \cos\left(at + \frac{\pi}{2}\alpha\right)$ (29)
4 . $D^\alpha\ (e^{kt}) = k^\alpha e^{kt}$ (30)
5 . $D^\alpha(C) = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)}$ (31)

2. Method to solving the Fractal Differential Equation

In this section, we will take some methods of solving fractal differential equations of the type Riemann-Liouville (homogeneous and non-homogeneous) with an explanation of the method in detail and solving an example for each method mentioned

2.1.Special method

It's the way we use the basic rules to derive the fractal differential equations mentioned in (1.6) and this method is considered one of the simplified methods for solving types of non-

homogeneous fractal differential equations, and to illustrate the method we will mention the examples below.
Example
Solving the fractal differential equation $D^{\frac{1}{2}}y = (t^2 - 2t), \quad \alpha = \frac{1}{2}, \quad y(0) = 0$

$$y = D^{\frac{1}{2}}t^2 - D^{\frac{1}{2}}2t$$
$$= \frac{\Gamma(2+1)}{\Gamma(2+1-\frac{1}{2})} t^{\frac{3}{2}} - 2\frac{\Gamma(2)}{\Gamma\left(1+1-\frac{1}{2}\right)} t^{\frac{1}{2}}$$
$$= \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)} t^{\frac{3}{2}} - 2\frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)} t^{\frac{1}{2}}$$
$$= \frac{16}{15\sqrt{\pi}} t^{\frac{5}{2}} - \frac{8}{\sqrt{\pi}} t^{\frac{3}{2}}$$

2.2 . Laplace transform Method

In this section, we present the Laplace transform of the Riemann–Liouville for fractional derivatives $D^{\alpha}y$

Definition (4): Defined the Laplace transform of $y(t)$ as [5]

$$L\, y(t) = \int_0^{\infty} e^{-st}\ y(t)dt \tag{32}$$
$$= \lim_{b\to\infty} \int_0^{b} e^{-st} y(t)dt = Y(s) \tag{33}$$

Sometimes it is convenient to denote the Laplace transform of $y(t)$ by $Y(s)$.

Laplace Transform of Riemann –Liouville Fractional Derivative:
Lemma(2), [34]

Let $Re(\alpha) > 0$ and $n = [Re(\alpha)] + 1; y \in AC^n [0, b]$, for any b > 0
Also, let the following estimate

$$|y(t)| \leq Be^{qt} \tag{34}$$

hold, for constants $B > 0$ and $q > 0$, and if $y^k(0) = 0, k = (0,1,2, \ldots, n-1)$ then the relation

$$L(D^{\alpha}y)(s) = s^{\alpha}(L\, y(s)) \tag{35}$$

is valid for $Re(s) > q$

properties of Laplace transform [3]

Here are some useful properties of Laplace transforms that are applied in solving fractal differential equations for $Y(s) = L\,[y(t)]$ and $G(s) = L\,[g(t)]$ (36)

1. $L[y(t) + g(t)] = Y(s) + G(s)$ (37)

2. $L[ay(t) + bg(t)] = aY(s) + b\,G(s)$ (38)

3. $L[t^{\alpha}] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \qquad \alpha \geq 0$ (39)

4. $L\left[y^{(n)}(t)\right] = s^nY(s) - s^{n-1}y(o) - s^{n-2}y'(0) \ldots \ldots - y^{n-1}(0)$ (40)

5. $L[t^{\alpha}y(t)] = (-1)^n\, y^{(n)}(s)$ (41)

6 . $L(\int_0^{x} y(t)) = \frac{1}{s}\, Y(s)$ (42)

7 . $L[\int_0^x y(t).g(x-t)dt] = Y(s).G(S)$ (43)

Example

Solving the fractal differential equation $D^2y(t) + D^{\frac{3}{2}}y(t) + y(t) = 1 + t$ and $y'(0) = y(0) = 1$

Solution

Using Laplace transform[26]

$$s^2Y(s) - sy(0) - y'(0) + \frac{s^2Y(s) - sy(0) - y'(0)}{s^{\frac{1}{2}}} + Y(s) = \frac{1}{s} + \frac{1}{s^2}$$

$$\mathrm{Y(s)}\left(s^2 + s^{\frac{3}{2}} + 1\right) = \left(\frac{1}{s} + \frac{1}{s^2}\right)(s^2 + s^{\frac{3}{2}} + 1)$$

$$\mathrm{Y(s)} = \frac{1}{s} + \frac{1}{s^2}$$

using the inverse Laplace transform the exact solution is

$$y(t) = 1 + t$$

2.3. Inverse Fractional Shehu Transform Method [28]

It is a new method called the inverse Fractional Shehu Transformation method to solve the homogeneous and non-homogeneous linear differential equations and the Riemann-Liouville fractal differential

Definition(6),[28]

The Shehu transform of the function f(t) of exponential order is defined over the set of functions

.

$$A = \left\{\frac{f(t)}{\exists N}, n_1, n_2 > 0, |f(t)| < N \exp\left(\frac{|t|}{n!}\right) if\ t \in (-1)^i \times [0,\infty)\right\} \quad (44)$$

i.e

$$s\big(f(t)\big) = F(s,u) = \int_0^\infty exp\left(\frac{-st}{u}\right) f(t)dt, t > 0 \quad (45)$$

Some Propties of Shehu Transforms[28]

1. $s[\lambda f(t) \pm \mu g(t)] = \lambda s[f(t)] \pm \mu s[g(t)]$ (46)

2. $s\left[f^{(n)}(t)\right] = \frac{s^n}{u^n} f(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-(k+1)} f^{(k)}(0)$ (47)

3. $s[(f * g)(i)] = f(s,u).G(s,u)$ (48)

$$(f * g)(t) = \int_0^t f(\xi)g(t-\xi)d\xi = \int_0^t f(t-\xi)g(\xi)d\xi$$

Some Special Shehu Transforms are[28]

1. $s(1) = \frac{u}{s}$ (49)

2. $s(t) = \frac{u^2}{s^2}$ (50)

3. $s\left(\frac{t^n}{n!}\right) = \left(\frac{u}{s}\right)^{n+1}, n = 0,1,2,3, \dots \dots$ (51)

4. $s(t^{\alpha}) = \left(\frac{u}{s}\right)^{\alpha+1} f(\alpha+1)$ (52)

Inverse Shehu Transform:[28]

The inverse shehu transform function is

$$f(t) = s^{-1}[f(s,u)] \quad (53)$$

*Theorem(10), [*28]**:**

If $F(s,u)$ is the shehu transform of $f(t)$, then the shehu transform of Riemann Linville fractional integral for the function $f(t)$ of order α , is given by

$$s[I^{\alpha}f(t)] = \left(\frac{u}{s}\right)^{\alpha} f(s,u) \quad (54)$$

Theorem(11),[28]:

Let $n \in N$ and $\alpha > 0$ such that $n-1 < \alpha \leq n$,and $F(s,u)$ be the shehu transform of funaction $f(t)$,then the shehu transform denoted by $F(s,u)$of the Riemann –lioville fractional derivative of $f(t)$ of order α ,is given by

$$s[D^{\alpha}f(t)] = f(s,u) = \frac{s^{\alpha}}{u^{\alpha}} F(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{k} [D^{\alpha-k-1}f(t)]_{t=0} \quad (55)$$

Example(14),[28]

Consider the initial value of non-homogeneous

$y''(t) + D^{\frac{3}{2}}y(t) + y(t) = 1 + t$ and the initial condition

$$y(0) = y'(0) = 1$$

Apply the shehu transform

$$\frac{s^2}{u^2} y(s,u) - \frac{s}{u}y(0) - 1 + \frac{s^{\frac{3}{2}}}{u^{\frac{3}{2}}} y(s,u) - \frac{s^{\frac{1}{2}}}{u^{\frac{1}{2}}}y(0) - \frac{s^{-\frac{1}{2}}}{u^{-\frac{1}{2}}}y'(0) + y(s,u) = \frac{u}{s} + \frac{u^2}{s^2}$$

$$y(s,u)\left(\frac{s^2}{u^2} + \frac{s^{\frac{3}{2}}}{u^{\frac{3}{2}}} + 1\right) - \frac{s}{u} - 1 - \frac{s^{\frac{1}{2}}}{u^{\frac{1}{2}}} - \frac{s^{-\frac{1}{2}}}{u^{-\frac{1}{2}}} = \frac{u}{s} + \frac{u^2}{s^2}$$

$$y(s,u)\left(\frac{s^2}{u^2} + \frac{s^{\frac{3}{2}}}{u^{\frac{3}{2}}} + 1\right) = \frac{s}{u} + 1 + \frac{s^{\frac{1}{2}}}{u^{\frac{1}{2}}} + \frac{s^{-\frac{1}{2}}}{u^{-\frac{1}{2}}} + \frac{u}{s} + \frac{u^2}{s^2}$$

$$y(s,u)\left(\frac{s^2}{u^2} + \frac{s^{\frac{3}{2}}}{u^{\frac{3}{2}}} + 1\right) = \left(\frac{s^2}{u^2} + \frac{s^{\frac{3}{2}}}{u^{\frac{3}{2}}} + 1\right)\left(\frac{u}{s} + \frac{u^2}{s^2}\right)$$

$$y(s,u) = s\big(y(t)\big) = \frac{u}{s} + \frac{u^2}{s^2}$$

taking the inverse of the Shehu transform

$$y(t) = 1 + t$$

2.4 The Power Series Method [9]

The Power series is a fundamental tool in the study of elementary functions. They have been widely used in computational science for easily obtaining an approximation of functions. In thermal physics and many other sciences, this power expansion has allowed scientists to make an approximate study of many differential equations

Definition(7), [19] ,[23] A power series representation of the form

$$y(t) = \sum_{n=0}^{\infty} c_n (t - t_{\circ})^{n\alpha} = c_{\circ} + c_1(t - t_{\circ})^{\alpha} + c_2(t - t_{\circ})^{2\alpha} + (n\ time) \quad (56)$$

Where $0 \leq m-1 < \alpha \leq m, m \in N^+ and\ t \geq t_\circ$ is called a fractional power series about $t_\circ$ where t is variable and c_n are the coefficients of series.

Theorem(4), [23] We have the following two cases for the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}, t \geq 0$

(1) If the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ converges when $t = b > 0$, then it converges whenever $0 \leq t < b$.

(2) If the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ diverges when $t = d > 0$, then it diverges whenever $t > d$.

Theorem(5), [23]

For the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}, t \geq 0$ there are only three possibilities:

(1) The series converges only when $t = 0$.

(2) The series converges for each $t \geq 0$.

(3) There is a positive real number R such that the series converges whenever $0 \leq t < R$ and diverges whenever $t \geq R$.

Theorem (6), [19]

Suppose that the fractional power series $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ has a radius of convergence $R > 0$. If $f(t)$ is a function defined by $f(t) = \sum_{n=0}^{\infty} c_n t^{n\alpha}$ on $0 \leq t \leq R$,Then for $m-1 < \alpha \leq m$ and we have:

$$D^\alpha f(t) = \sum_{n=0}^{\infty} cn \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha-\alpha+1)} t^{(n-1)\alpha} \tag{57}$$

$$I^\alpha f(t) = \sum_{n=0}^{\infty} cn \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+\alpha+1)} t^{(n+1)\alpha} \tag{58}$$

Proof

Define $g(x) = \sum_{n=0}^{\infty} c_n x^n$ for $0 \leq x < R^\alpha$, where R^α is the convergence then:

$$D^\alpha g(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} g^{(m)}(\tau) d\tau$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} \left(\frac{d^m}{d\tau^m} \sum_{n=0}^{\infty} c_n \tau^n \right) d\tau$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} \left(\sum_{n=0}^{\infty} c_n \frac{d^m}{d\tau^m} \tau^n \right) d\tau$$

$$= \sum_{n=0}^{\infty} c_n \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} \left(\frac{d^m}{d\tau^m} \tau^n \right) d\tau$$

$$= \sum_{n=0}^{\infty} c_n D^\alpha (x^n) \tag{59}$$

For the Remaining part, considering the definition of $g(x)$ above one conclude that :

$$I^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} g(\tau) d\tau$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \left(\sum_{n=0}^{\infty} c_n \tau^n \right) d\tau$$

$$= \sum_{n=0}^{\infty} c_n \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \tau^n d\tau$$

$$= \sum_{n=0}^{\infty} c_n I^\alpha x^n \tag{60}$$

Example(10), [19] Solving the fractal differential equation

$D^2y(t) + D^{\frac{1}{2}}y(t) + y(t) = 8$ and $y'(0) = y(0) = 0$ $,0 < t, \alpha = \frac{1}{2}$

Solution

Using the FPS technique and considering formula(1.51) the solution of Equations can be written as:

$$y(t) = \sum_{n=0}^{\infty} c_n \, t^{\frac{n}{2}}$$

To complete the formulation of the FPS technique, we must compute the functions $D_0^1 \, y(t), D_0^{\frac{1}{2}}y(t)$ *and* $D_0^2y(t)$, However, the forms of these functions are given, respectively, as follows:

$$D_0^{\frac{1}{2}}y(t) = \sum_{n=0}^{\infty} c_{n+1} \frac{\Gamma(\frac{n+1}{2}+1)}{\Gamma(\frac{n}{2}+1)} \, t^{\frac{n}{2}}$$

$$D_0^1y(t) = c_1t^{-\frac{1}{2}} + c_2 + \sum_{n=3}^{\infty} c_n \frac{n}{2} \, t^{\frac{n-2}{2}}$$

$$D_0^2y(t) = -\frac{1}{2}c_1t^{-\frac{3}{2}} + \frac{3}{4}c_3t^{-\frac{3}{2}} + \sum_{n=4}^{\infty} c_n \frac{n}{2}(\frac{n}{2}-1) \, t^{\frac{n-4}{2}}$$

But since $\{t|t \geq 0\}$ is the domain of the solution, then the values of the coefficients c_1 and c_3 must be zeros. On the other aspect as well, the substituting the initial conditions into an Equation of power series We will get $c_\circ, c_2$ Therefore the discretized form of the functions $y(t), D_0^{\frac{1}{2}}y(t)$ *and* $D_0^2y(t)$, is obtained. The resulting new form will be as follows:

$$y(t) = \sum_{n=0}^{\infty} c_n \, t^{\frac{n}{2}}$$

$$D_0^{\frac{1}{2}}y(t) = c_4 \frac{2}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} + \sum_{n=4}^{\infty} c_{n+1} \frac{\Gamma(\frac{n+1}{2}+1)}{\Gamma(\frac{n}{2}+1)} \, t^{\frac{n}{2}}$$

$$D_0^2y(t) = 2\,c_4 + \frac{15}{4}c_5t^{\frac{1}{2}} + \frac{35}{4}c_7t^{\frac{3}{2}} + \sum_{n=4}^{\infty} c_{n+4} \frac{n+4}{2}(\frac{n+4}{2}-1) \, t^{\frac{n}{2}}$$

we then will obtain recursively the following

$c_4 = 4\,, c_5 = 0, c_6 = 0\,, c_7 = \dfrac{-128}{105\sqrt{\pi}}$ *and*

$$c_{n+4} = \frac{-4}{(n+2)(n+4)}(c_n + c_{n+1} \frac{\left(\frac{n+1}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1\right)}), n \geq 4$$

So, the 15th-truncated series approximation of $y(t)$ is

$$y_{15}(t) = 4t^2 - \frac{128}{105\sqrt{\pi}}t^{\frac{7}{2}} - \frac{1}{3} \, t^4 + \frac{1}{15}t^5 + \frac{1024}{1039\sqrt{\pi}} \, t^{\frac{11}{2}} + \frac{1}{190}t^6 - \frac{1024}{135135\sqrt{\pi}} \, t^{\frac{13}{2}}$$

The FPS technique has the advantage that it is possible to pick any point in the interval of integration and as well the approximate solution and all its derivatives will be applicable. In other words, a continuous approximate solution will be obtained.

2.5. The generalized Mittag-Leffler method [4]

The Mittag-Leffler (1902–1905) functions E_α and $E_{\alpha,\beta}$ defined by the power series

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)} \quad , E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)} \quad \alpha > 0, \beta > 0 \tag{61}$$

have already proved their efficiency as solutions of fractional order differential and integral equations and thus have become important elements of the fractional calculus theory and applications, we will explain how to solve some differential equations with fractional level through the imposition of the generalized Mittag- Leffler function $E_\alpha(z)$. The generalized Mittag-Leffler method suggests that the linear term $y(t)$ is decomposed by an infinite series of components, and it also solves homogeneous fractal differential equations

$$y = E_\alpha(at^\alpha) = \sum_{n=0}^{\infty} \frac{c^n t^{\alpha n}}{\Gamma(n\alpha+1)} \tag{62}$$

We will use the following definitions of fractional calculus:

$$D^\alpha y(t) = \sum_{n=0}^{\infty} c^n \frac{t^{(n-1)\alpha}}{\Gamma(n\alpha-\alpha+1)} \tag{63}$$

$$D^{2\alpha} y(t) = \sum_{n=0}^{\infty} c^n \frac{t^{(n-2)\alpha}}{\Gamma((n-2)\alpha+1)} \tag{64}$$

And the solution approximation is

$$y(t) = c_\circ + \sum_{n=1}^{\infty} \frac{c^n t^{\alpha n}}{\Gamma(n\alpha+1)} \tag{65}$$

We will present two examples given below as an illustration of the method discussed above for solving fractional differential equations.

Example

Solving the fractal differential $D^\alpha\, y - y = 0$

Solution: Suppose that $y = \sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}$ and We substitute it in the given equation.[33]

$$let\ y = \sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}$$

$$D^\alpha\left(\sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}\right) - \left(\sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}\right) = 0$$

$$\left(\sum_{n=0}^{\infty} \frac{c^n}{\Gamma(n\alpha+1)} D^\alpha x^{n\alpha}\right) - \left(\sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}\right) = 0$$

$$\left(\sum_{n=0}^{\infty} \frac{c^n}{\Gamma(n\alpha+1)} \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha-\alpha+1)}\, x^{n\alpha}\right) - \left(\sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}\right) = 0$$

$$\left(\sum_{n=0}^{\infty} \frac{c^n}{\Gamma(\alpha(n-1)+1)}\, x^{n\alpha}\right) - \left(\sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}\right) = 0$$

Let $k = n - 1$ $\qquad\qquad$ $k = n$

$$\left(\sum_{n=0}^{\infty} \frac{c^{k+1}}{\Gamma(k\alpha+1)}\, x^{k\alpha}\right) - \left(\sum_{n=0}^{\infty} \frac{c^k x^{k\alpha}}{\Gamma(k\alpha+1)}\right) = 0$$

$$\sum_{n=0}^{\infty} c^{k+1} - c^k \left(\frac{x^{\alpha k}}{\Gamma(\alpha k+1)}\right) = 0 \quad and \quad \frac{x^{\alpha k}}{\Gamma(\alpha k+1)} \neq 0$$

$$c^{k+1} - c^k = 0 \Rightarrow c^k(c-1) = 0 \; but \; c^k \neq 0 \Rightarrow c-1 = 0 \; \because c = 1$$

$$y(x) = c^{\circ} + c^1 \frac{x^{\alpha}}{\Gamma(\alpha+1)} + c^2 \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + c^3 \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots$$

The general solution is $y = \sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}$.

2.6. Spline Interpolation Techniques for Approximating the Solution of Fractional

Definition A function $S(t)$ is called natural G-spline for the knots $t_1, t_2, \ldots, t_K$ and the matrix E^* of order m provided that it satisfies the following conditions:

1. $S(t) \in \Pi_{2m-1}$ in $(t_i, t_{i+1}), i = 1, 2, \ldots, k-1$ (66)
2. $S(t) \in \Pi_{m-1}$ in $(-\infty, t_1)$and in (t_k, ∞). (67)
3. $S(t) \in c^{m-1}$ $(-\infty, \infty)$. (68)
4. If $a^*_{ij} = 0$, then $S^{(2m-j-1)}(t)$is continuous at $t = t_i$ (69)

Let S $(E^*;\ t_1, t_2, \ldots, t_K)$ denotes the class of all G-spline of order m.

The definition of the fractional derivatives and some well-known results of fractional calculus tells us that we interpret fractional differential equations such as, [Oldham, 1998]:

$$D^{\alpha}\, y(t) = y(t, y(t)), \qquad y(a) = m \qquad (70)$$

where $n < \alpha < n+1$ and $D^{\alpha} := \frac{d^{\,\alpha}}{dx}$. Hence, upon carrying D^{1-q} to both sides of (6), yields:

$$D^{1-\alpha}\, D^{\,\alpha}\, y(t) = D^{1-\alpha}\; Y(t, y(t)), \qquad y(a) = m \qquad (71)$$

with $n < \alpha < n+1, n \in R$

Equation (7) can be simplified using formulas and definitions of the fractional derivatives, to get:

$$y'(t) = g(t,y), y(a) = m \;\; , x \in [a,b] \qquad (72)$$

The exact solution of (72) evaluated at $tk = a + kh$, as:

$$Y(t_k) - Y(t_\ell) = \int_{t_\ell}^{t_k} t\; g(t,y)dt \quad ,0 \leq \ell \leq k \qquad (73)$$

and then replacing g with its G-spline Interpolant. An m-th order linear multistep formula of the general type can be given by, [46]

$$y_{n+k} - y_{n+\ell} = \sum_{j=0}^{p} \sum_{i=0}^{k} B_{ij}\, b^{j+1} g^{(i)}(t_{n+i}\, , y_{n+i}) \qquad (73)$$

where y_j is an approximation to $Y(t_j)$ and $[t_n, t_{n+k}] \subset [a,b]$.

Now, we pick k and α along with the m −poised HB-problem corresponding to the n values:

$$\{\ \varphi_i^{(j)} = \varphi^{(j)}\ (i), (i,j) \in e\}$$

where $\varphi(s) \equiv g(t_n + sh, Y(t_n + sh)), for\ 0 \leq s \leq k$

Then as we mentioned previously in section two, the G −spline interpolant to φ can be given in terms of the fundamental G −splines $L_{ij}(t)$, by:

$$S_m(s) = \sum_{(i,j)\in e} L_{ij}\ (\ s)\ \varphi_i^{(j)} \qquad (74)$$

Referring again to our HB problem, there is a unique G −spline S_m in
$S(E^*;\ t_1, t_2, \dots, t_k)$,
such that $S_m^{(j)}(i) = \varphi_i^{(j)}$
To determine the coefficients β_{ij} in (73) we replace g in (74) by its G-spline interpolate, make a change of variables, integrate and compare the results with (74).
As a consequence of the uniqueness of the G-spline interpolant, and the sense of theorem (1) it follows that
$\beta_{ij} = \int_{\ell}^{k} L_{ij}\,(s)ds$ *Lij(s) ds* (75)
Finally, it is the time to summarize the above results as follows:
Equation (10) can have an approximate solution using linear multistep methods in terms of G-spline interpolation, as follows:
$y_{n+k} - y_{n+\ell} = \sum_{j=0}^{p} \sum_{i=0}^{k} B_{ij}\, b^{j+1} g^{(i)}(t_{n+i}\,, y_{n+i})$ (76)
where B_{ij} are presented in equation (75).
Next, we give one example as an illustration of the above-discussed approach for solving fractional differential equations. The obtained results are compared with the exact solution which is available in this one example
Example
Consider the fractional differential equation

$$y^{(1/2)}\,(t) = -y(t) \;+\; t^2 \;+ \frac{2\,x^{3/2}}{\Gamma(5/2)}$$

$$y(0) = 0$$

where the exact solution is given by $y(t) = t^2$.

Consider it is required that a three-step method be constructed in such a way that it is exact for $y \in \Pi_4$
To construct such a method via G-splines, an HB problem must be first chosen. The choice:

$$\Delta = \{0, 1, 2\}$$

are taken to be the knote points and let: $e = \{(0,0), (0,1), (1,0), (1,1), (2,0)\}$
We shall seek for $S_4(s) \in S_4(E_*, \Delta)$, with:

$$\mathrm{E} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and for which:

$$S_4(s) = \varphi^{(j)}\,(i), \qquad (i, j\,) \in e$$

Integrating $S_4(s)$ over $[1, 2]$ yields the closed formula:

$$y_{n+2} = y_{n+1} + [h\{\beta_{00}g(t_n, y_n) + \beta_{10}g(t_{n+1}, y_{n+1}) + \beta_{20}g(t_{n+2}, y_{n+2})\} + h_2\,\{\beta_{01}g'\,(t_n, y_n) + \beta_{11}g'\,(t_{n+1}, y_{n+1})\}] \dots\dots\dots\dots\dots\dots (12)$$

$$\beta_{00} = \int_1^2 L_{00}\ (s)ds = \frac{503}{3072}\ ,\quad \beta_{10} = \int_1^2 L_{10}\ (s)ds = \frac{1748}{3072}\ ,\quad \beta_{20} = \int_1^2 L_{20}\ (s)ds = \frac{821}{3072}$$

$$\beta_{01} = \int_1^2 L_{01}\ (s)ds = \frac{150}{3072}\ , \beta_{02} = \int_1^2 L_{02}\ (s)ds = \frac{1068}{3072}$$

$$g(t, y) = D^{1/2}\left[-y + t^2 + \frac{2t^{3/2}}{\Gamma(\frac{5}{2})}\right]$$

$$= y(t) - t^2 + 2t$$

$g'(t, y) = y - t^2 + 2$

where:

$$L_{00}(s) = [128 - 494\,s^2 + 411\,s^3 + 25\,s_+^7 - 70\,s_+^6 - 20(s-1)_+^7 - 140(s-1)_+^6 - 5(s-2)_+^7]/128$$

$$L_{01}(s) = [64\,s - 150\,s^2 + 95\,s^3 + 5\,s_+^7 - 14\,s_+^6 - 4(s-1)_+^7 - 28(s-1)_+^6 - 5(s-2)_+^7]/64$$

$$L_{01}(s) = [118\,s - 95\,s^3 - 5\,s_+^7 - 14\,s_+^6 - 4(s-1)_+^7 - 28(s-1)_+^6 + (s-2)_+^7]/32$$

$$L_{11}(s) = [-45\,s^2 + 63\,s - 5\,s_+^7 - 14\,s_+^6 - 4(s-1)_+^7 - 28(s-1)_+^6 + (s-2)_+^7]/32$$

$$L_{20}(s) = [22\,s^2 - 31\,s^3 - 5\,s_+^7 - 14\,s_+^6 - 4(s-1)_+^7]/128$$

$$(s-a)_+^n = \begin{Bmatrix} (s-a)^n & \text{if } s > a \\ 0 & \text{if } s \le a \end{Bmatrix}$$

And the exact solution $Y(t) = t^2$

3. Conducting research and results

The quantitative component of the research placed a premium on data collection, processing, and analysis. In a survey conducted during the research, a nine-level Likert scale was used to assess respondents' perceptions and assessments of the dependent variable (transitional crisis), as well as the independent variables (heritage of socialism, geopolitics, nomenclature authorities, deficit of institutional changes, and neoliberal ideology). The dependent variable (transitional crisis) was quantified using a scale ranging from lowest (1) to most significant (5). (5). Concerning the independent factors, the negative influence on the dependent variable was quantified from a minimum of (1) to a maximum of (5). The study required respondents to complete 500 questions for each nation (Iraq, Syria, and Egypt), totaling 1,500 respondents. SPSS software was used to process the data collected for this investigation. For data analysis, correlation analysis, and multi-correlation, descriptive statistics were employed under the goal specified in the working hypothesis. A multiple linear regression model was used (using the least-squares approach) and a hierarchical multiple regression model.

Conclusion

In this paper, the definitions, concepts, and basics of linear and nonlinear fractional differential equations of the Riemann-Liouville type for both homogeneous and homogeneous types were studied, and some methods of solving fractional differential equations were presented with an explanation of each method and how to solve the equation in it. An example is taken for each method presented to show how to solve in this way. Among the methods presented are (Special method, Laplace transform method, inverse fractional Shehu Transform method, Power Series method, Generalized Mittag-Leffler method, and Spline Interpolation Techniques for Approximating the Solution of Fractional Differential Equations)This research aims to collect the basic concepts of fractional differential Riemann-Liouville equations and to present some direct and approximate methods for solving these equations and comparing the methods of solutions. We found that there are direct methods for solving and there are numerical methods for solving and approximate methods Laplace transform.

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